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BOOK REVIEW

C. M. ELLIOTT and J. R. OCKENDON, Weak and Variational Methods for Moving Boundary Problems. Pitman Advanced Publishing Program: Boston, London, Melbourne, 1982, 213 pp. £12.00.

MANY boundary-value problems of practical importance contain the special difficulty that a section of the boundary is unknown and must be determined as part of the solution. Such problems are usually referred to as free or moving boundary-value problems, depending on whether the boundaries are stationary or in motion. Recently there has been a growing tendency to use the term 'free boundary' to cover both types. However, the authors have taken the opposite stance in their title and their book includes both steady and time-dependent problems. More precisely, it is about two approaches to the solution of these problems which have been developed and studied extensively in the last 10 years. The attraction of these weak and variational methods is two-fold. The properties of existence and uniqueness have proved more susceptible to examination than is the case for the older, classical solutions; and numerical algorithms emerge which are more easily justified and, most importantly, avoid the need to track the unknown boundary as the solution proceeds. All this is because the problem is reformulated on a fixed domain and the boundary position is revealed *a posteriori* as a property of the solution. The authors are at pains to illustrate that problems which are physically diverse may conform to a uniform mathematical structure when cast in weak or variational form. In the latter case the relevance of the ideas and techniques of optimisation and mathematical programming is a useful bonus.

By way of introductory background, a number of free and moving boundary problems are formulated in classical terms. They include shock waves, frictional and contact problems in solid mechanics, problems in heat conduction, molecular diffusion and incompressible hydrodynamics, and seepage through porous media. The obstacle problem is used as an illustrative example several times in the book. An elastic string or membrane with a fixed boundary is in equilibrium under a small transverse displacement due to a pressure over the whole membrane but is constrained to lie on one side of a rigid obstacle with which part of the membrane is in contact.

Very briefly, because it is strictly outside their title, the authors mention alternate formulations leading to solution by integral equations, mapping, embedding, boundary-fixing changes of variable, finite-differences and finite-elements. Then comes a gentle introduction to the two methods which are the central theme of the book. The conservational requirement motivates the discussion of weak solutions for several amenable problems. The variational formulation is introduced through the obstacle problem and coupled with the all-important, linear complementarity statement. Transformations of the so-called Baiocchi type, which introduce the degree of continuity essential before a problem can be formulated in variational form, are explained through the Hele–Shaw problem. This problem and its relationship to others in porous flow and Stefan problems in supercooled liquids receives frequent mention.

From Chapter 3 onwards the treatment is more intense and in order to understand the precise mathematical ideas and arguments the reader must be familiar with the language and basic relevant theorems of functional analysis. An appendix contains the necessary seminar material. The analysis of the weak enthalpy method and its application to Stefan problems form Chapter 3. Chapter 4 deals similarly with elliptic and parabolic variational inequalities and includes four numerical examples. These two chapters form the heart of the book with rigorous attention to detail. The remainder of the book deals briefly with unresolved difficulties such as 'mushy' regions, other generalised Stefan and dam problems, and aspects of the equivalence between variational inequalities and weak formulations.

The book ends on a note of impartiality: open questions remain about the detection of problems ripe for formulation in weak or variational form and whether a practical numerical algorithm would emerge. The authors have performed a useful and timely function in pulling together many strands in this fast-developing subject and giving it a provisional coherence.

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